PMM U.S.S.R.,Vol.51,No.4,pp.456-462,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

OSCILLATIONS IN A CONTINUOUSLY STRATIFIED FLUID IN A MOVING VESSEL, AND THEIR CONTROL*

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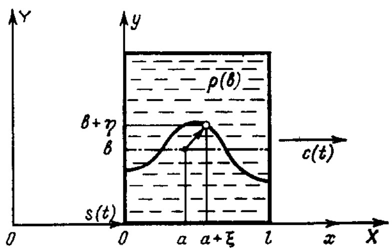
Classical models of a heavy ideal fluid are used to study internal wave motions of a stable stratified fluid in a moving vessel, and methods of controlling these oscillations. The case of exponential stratification, which differs in a number of essential features from the case of discrete stratification studied earlier in /1-3/, is considered.

1. Formulation of the problem and initial assumptions. We consider one-dimensional motion (along the OX axis) of a rectangular vessel filled with a heavy ideal incompressible fluid (see the figure). We assume that the density ρ of the fluid increases with depth (the fluid is permanently stratified), i.e. $\rho'(b) < 0$, where b is the vertical Lagrangian coordinate /4/. At the initial instant $t = 0$ the fluid is at rest relative to the walls of the vessel, and for $t > 0$ the vessel begins to move with an acceleration $w(t) = dc(t)/dt$ in the horizontal direction, e.g. to the right (see the figure). We require to find the wave motions in the fluid caused by the permanent stratification and generated by the motion of the vessel with a given variable velocity $c(t)$.



In order to describe the internal wave motions of the fluid, we shall use a moving oxy coordinate system attached to the left-hand wall of the vessel. Following /5/, we shall write the equations of hydrodynamics in terms of the Lagrange variables a, b which identify the fluid particles and are more suitable for further analysis. The equations of motion are assumed such, that the displacements of the fluid particles are two-dimensional. As a result of all these assumptions, we have the following non-linear Lagrange's equations for the coordinates $x = x(a, b, t)$, $y = y(a, b, t)$ of the fluid particles, in the moving coordinate system:

$$
\begin{aligned}
&(x^{\cdot\cdot} + w)\frac{\partial x}{\partial a} + (y^{\cdot\cdot} + g)\frac{\partial y}{\partial a} = -\frac{1}{\rho}\frac{\partial P}{\partial a} \\
&(x^{\cdot\cdot} + w)\frac{\partial x}{\partial b} + (y^{\cdot\cdot} + g)\frac{\partial y}{\partial b} = -\frac{1}{\rho}\frac{\partial P}{\partial b} \\
&\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a} = 1; \quad \rho = \rho(b)
\end{aligned}
\tag{1.1}
$$

Here g is the acceleration due to gravity, $\rho(b)$ is a known function of the density of the individual particle of the fluid, and a dot denotes a partial time derivative. In subsequent investigations it will be convenient to adopt, as the Lagrange variables (arguments) a, b, the initial positions of the fluid particles (see below).

The first two equations of system (1.1) represent Netwon's equations for fluid particles. The third equation expresses the condition of incompressibility of the fluid. The fourth equation (an assumption) means that the density of an individual fluid particle is preserved during the motion and depends only on the Lagrange variable b.

The independent Lagrange variables a, b, the coordinates x, y of the fluid particles (or Euler's variables) and the time t, vary within the stated limits

$$
\begin{aligned}
&(a, b) \in D = \{a, b\colon 0 \leqslant a \leqslant l,\ 0 \leqslant b \leqslant h\} \\
&(x, y) \in D = \{x, y\colon 0 \leqslant x \leqslant l,\ 0 \leqslant y \leqslant h\} \\
&0 \leqslant t \leqslant T,\ T < \infty
\end{aligned}
\tag{1.2}
$$

Here l is the length of the vessel, and h is the height of the fluid layer. In accordance with the assumptions made above, we take the initial conditions for the coordinates x, y in the form

*Prikl.Matem.Mekhan.,51,4,585-592,1987

$$x(a, b, 0) = a,\ y(a, b, 0) = b;\ x^{\cdot}(a, b, 0) = y^{\cdot}(a, b, 0) = 0 \tag{1.3}$$

and the conditions correspond to the state of the fluid at rest relative to the walls of the vessel at $t = 0$.

The boundary conditions mean that the walls of the vessel are impermeable

$$x(0, b, t) = y(a, 0, t) = 0,\ x(l, b, t) = l,\ y(a, h, t) = h \tag{1.4}$$

We further assume that the simplifying assumption that the acceleration of the vessel is fairly low, i.e. that $|w(t)|\,g^{-1} \ll 1$ for all $t \in [0, T]$, holds. This condition (see /1/) enables us to linearize the equations of hydrodynamics (1.1) with help of the following substitutions:

$$x = a + \xi,\ \xi = \xi(a, b, t);\ y = b + \eta,\ \eta = \eta(a, b, t) \tag{1.5}$$

$$P = -g\int_0^y \rho(b)\,db + H,\quad H = H(a, b, t)$$

The unknown quantities ξ, η in (1.5) represent the small displacements of the fluid particles from their initial positions (1.3), and H is a small deviation of the pressure from its hydrostatic value. The system of linearized equations of hydrodynamics takes the form

$$\begin{aligned} \xi^{\cdot\cdot} &= -\rho^{-1}\partial H/\partial a - w(t) \\ \eta^{\cdot\cdot} &= -\rho^{-1}\partial H/\partial b - N^2(b)\,\eta \\ \partial\xi/\partial a &+ \partial\eta/\partial b = 0;\ N^2(b) = -g\rho'(b)/\rho(b) \end{aligned} \tag{1.6}$$

The assumption corresponding to the simplest model of the stratified fluid states that the magnitude of the square of the Brunt-Vaisala frequency $N^2(b)$ /4/ is constant, i.e.

$$N^2(b) \equiv N_0^2 = \text{const} > 0 \tag{1.7}$$

In this case the fluid density varies exponentially

$$\rho(b) = \rho_0 \exp(-N_0^2 b/g),\ \rho_0 = \rho(0) \tag{1.8}$$

Henceforth, we will assume that assumption (1.7) or (1.8) holds with a sufficient degree of accuracy. Then the solution of system (1.6), (1.8) will be obtained in analytic form. Below we shall also discuss the properties of the solution in the more general case when the quantity $N^2(b) \geqslant C > 0$ is not constant, but varies with depth.

The initial and boundary conditions for the new unknown variables ξ, η are obtained from the conditions (1.3), (1.4) and the substitution formulas (1.5), and remain homogeneous.

We further introduce the function $\psi = \psi(a, b, t)$ (the stream function of an inhomogeneous fluid) such, that

$$\xi = \partial\psi/\partial b,\ \eta = -\partial\psi/\partial a \tag{1.9}$$

Then the third equation of system (1.6) will be satisfied identically. Using relations (1.9), we can eliminate the unknown function H from the first two equations of (1.6), and obtain a single equation for the stream function ψ. To make the solution of the corresponding boundary value problem easier, we introduce the independent dimensionless variables, i.e. the Lagrange variables α, β and time τ, the parameters of the system r and γ, and the stream function $\Psi = \Psi(\alpha, \beta, \tau)$. The substitution formulas have the form

$$\begin{aligned} &\alpha = a/h,\ \beta = b/h;\ \tau = N_0 t,\ 0 \leqslant \tau \leqslant \Theta = N_0 T \\ &(\alpha, \beta) \in \Delta = \{\alpha, \beta\colon 0 \leqslant \alpha \leqslant r,\ 0 \leqslant \beta \leqslant 1\} \\ &r = l/h,\ \gamma^2 = N_0^2 h/g,\ \Psi = \psi/h^2 \end{aligned} \tag{1.10}$$

Here Δ denotes a rectangular domain of variation of the dimensionless Lagrange variables α, β. According to (1.6)-(1.10) the equation for the dimensionless stream function can be written in the form

$$\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial^2\Psi}{\partial\alpha^2} + \frac{\partial^2\Psi}{\partial\beta^2}\right) - \gamma\frac{\partial^3\Psi}{\partial\tau^2\partial\beta} + \frac{\partial^2\Psi}{\partial\alpha^2} = \gamma w_*(\tau) \tag{1.11}$$

$$w_*(\tau) \equiv (N_0^2 h)^{-1}\, w(\tau/N_0)$$

The initial and boundary conditions for the stream function $\Psi(\alpha, \beta, \tau)$ follows from relations (1.3)-(1.5) and (1.9), and have the form

$$\Psi(\alpha, \beta, 0) = \frac{\partial\Psi}{\partial\tau}(\alpha, \beta, 0) = 0 \tag{1.12}$$

$$\Psi(0, \beta, \tau) = \Psi(r, \beta, \tau) = \Psi(\alpha, 0, \tau) = \Psi(\alpha, 1, \tau) = 0$$

Thus we require to find a solution of the boundary value problem (1.11), (1.12).

2. Formal solution of the internal boundary value problem. Applying the method of separation of variables /1/ (the Fourier method) to the homogeneous Eq.(1.11), we can find the eigenfunctions of the corresponding boundary value problem in spatial variables α, β

$$\Phi_{mn}(\alpha, \beta) = \sin(\pi r^{-1} m\alpha) \sin(\pi n\beta) \exp(^1/_2\gamma\beta) \quad (2.1)$$
$$(\alpha, \beta) \in \Delta, \; m, n = 1, 2, \ldots$$

The system of eigenfunctions $\{\Phi_{mn}\}$ (2.1) is complete and orthogonal, with weight $e^{-\gamma\beta}$ in the region $(\alpha, \beta) \in \Delta$, and this can be confirmed directly.

Analysing the equations for the Fourier coefficients depending on time τ, we arrive at the following expressions for the eigenfrequencies of the small free oscillations of an exponentially stratified fluid completely filling a rectangular vessel:

$$\omega_{mn}^2 = \frac{m^2}{r^2}\left(\frac{m^2}{r^2} + n^2 + \frac{\gamma^2}{4\pi^2}\right)^{-1}, \quad m, n = 1, 2, \ldots \quad (2.2)$$

It should be noted that the spectrum of the eigenfrequencies $\{\omega_{mn}\}$ is discrete, and unlike the case of a discretely stratified fluid (see /1-3/ et al.), it depends on two independent indices $m, n \geqslant 1$. Moreover, the frequencies are bounded: $0 < \omega_{mn} < 1$ (in dimensionless variables the Brunt-Vaisala frequency N_0 forms their upper limit). It is also interesting to note that the set of numbers $\{\omega_{mn}\}$ is dense everywhere on the segment $\omega \in [0, 1]$, i.e. for any $0 < \gamma^2, r^2 < \infty$ there exist natural $m, n > 1$ such that $|\omega_{mn} - \omega| \leqslant \varepsilon$ where $\varepsilon > 0$ is arbitrarily small. The corresponding values of the indices m and n depend, naturally, on ε and also on $\gamma^2, r^2 > 0$. This means that in an arbitrarily small ε-neighbourhood of any given $\omega \in [0, 1]$, we can find a denumerable number of frequencies ω_{mn} and a denumerable number of sequences converging to an arbitrary value of $\omega \in [0, 1]$.

In the case when $N^2(b) \not\equiv N_0^2$ $(N^2(b) \geqslant C > 0)$, the eigenfunctions $\Phi_{mn}(\alpha, \beta)$ cannot, in general, be represented by explicit analytic expressions allowing a complete analysis. It can, however, be established that there exists, as before, a bounded discrete spectrum of eigenfrequencies $0 < \omega_{mn} < 1$ (in dimensional variables the spectrum has the maximum Brunt-Vaisala frequency N^*, $N^* = \max_b N(b)$, $\tau = N^* t$) as the upper limit). The property that the set $\{\omega_{mn}\}$ is dense everywhere on the segment $\omega \in [0, 1]$ is also preserved; the above assertions follow from a analysis of the corresponding Sturm-Liouville problem.

The properties of the spectrum of natural oscillations of a continuously stratified fluid completely filling the vessel, radically change the character of the behaviour of the solution of the inhomogeneous problem (1.11), (1.12) as compared with the corresponding cases of discrete stratification /1-3/ for which the spectrum of $\{\omega_n\}$ is one-dimensional and increases monotonically with n, and $\omega_n \sim \sqrt{n}$ for $n \gg 1$.

Thus, applying the Fourier method and using the complete orthogonal system of the eigenfunctions $\{\Phi_{mn}(\alpha, \beta)\}$ (2.1) obtained above, we can construct the required solution of problem (1.11), (1.12) in the form of a binary series

$$\Psi(\alpha, \beta, \tau) = \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \Psi_{in} \Phi_{2i+1, n}(\alpha, \beta) \theta_{in}(\tau) \quad (2.3)$$
$$\Psi_{in} = \Psi_{in}(r, \gamma) = -8\pi^{-2}\gamma [1 - (-1)^n e^{-1/2\gamma}] \chi_{in}(r, \gamma)$$
$$\chi_{in}(r, \gamma) = n \left\{(2i+1)\left(n^2 + \frac{\gamma^2}{4\pi^2}\right)\left[\frac{(2i+1)^2}{r^2} + n^2 + \frac{\gamma^2}{4\pi^2}\right]\right\}^{-1}$$
$$\theta_{in}(\tau) = \int_0^\tau w_*(\tau') \frac{\sin \omega_{2i+1, n}(\tau - \tau')}{\omega_{2i+1, n}} d\tau' =$$
$$\int_0^\tau c_*(\tau') \cos \omega_{2i+1, n}(\tau - \tau') d\tau' - c_*(0) \frac{\sin \omega_{2i+1, n}\tau}{\omega_{2i+1, n}}$$
$$c_*(\tau) = (N_0 h)^{-1} c(\tau / N_0)$$

It should be noted that $\Psi \sim \gamma$ where the coefficient $\gamma > 0$ characterizes the relative stratification. The term in (2.3) corresponding to even values of $m = 2i$, do not appear, just as in the cases of discrete stratification /1-3/. The small displacements of the fluid particles $\xi(a, b, t)$, $\eta(a, b, t)$ (1.5) relative to the initial position $(a, b) \in D$ are found using formulas (1.9), (1.10), with the known expression (2.3) used as the dimensionless stream function $\Psi(\alpha, \beta, \tau)$.

Let the relative stratification be small, i.e. the ratio $\gamma = N_0/\sqrt{g/h} \ll 1$. Then neglecting terms of order γ^2 and higher, we can obtain simpler expressions for the quantities Ψ, ξ, η sought. The summation in (2.3) will be carried out over odd values of the index $n = 2j + 1$, $j = 0, 1, \ldots$ The expressions for the coefficients Ψ_{in} (2.3) and functions $\Phi_{mn}(\alpha, \beta)$ (2.1) are reduced to the form $(i, j \geqslant 0)$:

$$\Psi_{in} = \Psi_{in}^* (r, \gamma) = -16\pi^{-2}\gamma \{(2i+1)(2j+1)[(2i+1)^2 r^{-2} + (2j+1)^2]\}^{-1} \tag{2.4}$$
$$\Phi_{mn}(\alpha, \beta) = \Phi_{ij}^*(\alpha, \beta) = \sin \pi r^{-1}(2i+1)\alpha \sin \pi(2j+1)\beta$$

The expressions for the derivatives of the functions $\Phi_{mn}(\alpha, \beta)$ in α, β are also simplified, as well as the expressions for the displacements of the particles $\xi(a, b, t)$, $\eta(a, b, t)$. The eigenfrequencies ω_{mn} (2.2) can be expressed as functions of the stratification parameter γ more simply by neglecting terms of the order of $O(\gamma^2)$. The quantities of the order of $O(\gamma^2\tau)$ can be neglected in the functions $\theta_{in}(\tau)$ (2.3) containing $\omega_{mn}\tau$, provided that the quantity $\Theta = N_0 T$ is not very large. Thus for small $\gamma > 0$ the functions Ψ, ξ, η will be of the order of γ and will vanish for $\gamma = 0$, this obviously following from (1.11), (1.9). Moreover, the functions $\Psi = \xi = \eta \equiv 0$ of the acceleration $w(t) \equiv 0$ at $t \in [0, T]$.

The velocities of fluid particles $v_x(a, b, t)$, $v_y(a, b, t)$ are obtained in the linear approximation discussed here, from expressions (1.5), using formulas (1.9), (1.10) and (2.3) determining the displacements $\xi(a, b, t)$, $\eta(a, b, t)$, by differentiating the Fourier coefficients $\theta_{in}(\tau)$ in t

$$v_x(a, b, t) = \frac{\partial \xi}{\partial t} = h \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \Psi_{in} \frac{\partial \Phi_{2i+1,n}}{\partial \beta} \frac{d\theta_{in}}{dt} \tag{2.5}$$

$$v_y(a, b, t) = \frac{\partial \eta}{\partial t} = -h \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \Psi_{in} \frac{\partial \Phi_{2i+1,n}}{\partial \alpha} \frac{d\theta_{in}}{dt}$$

$$\frac{d\theta_{in}(\tau)}{d\tau} = N_0^{-1} \frac{d\theta_{in}(\tau)}{dt} = \int_0^{\tau} w_*(\tau') \cos \omega_{2i+1,n}(\tau - \tau')\, d\tau' =$$

$$-\omega_{2i+1,n} \int_0^{\tau} c_*(\tau') \sin \omega_{2i+1,n}(\tau - \tau')\, d\tau' + c_*(\tau) + c_*(0) \sin \omega_{2i+1,n}\tau$$

Thus the dual series (2.3)-(2.5) constructed provide a formal solution of the boundary value problem (1.11), (1.12) in dimensionless variables. The return to the initial dimensional variables is made using formulas (1.10). The convergence properties of the formal series obtained are studied below.

3. Investigating the solution of the internal boundary value problem. Using the majorant criteria of convergence of dual series we can establish that the series (2.3) for the stream function $\psi(\alpha, \beta, \tau)$ converges absolutely and uniformly in the region $(\alpha, \beta, \tau) \in \Delta \times [0, \Theta]$ (1.10) for the Riemann-integrable functions, and in particular for the piecewise smooth functions $w_*(\tau)$, $\tau \in [0, \Theta]$. Moreover, the quantity w_* will, in accordance with (1.5), be assumed to be sufficiently smooth so that the basic condition of applicability of the linear theory is satisfied.

The stream function Ψ is majorized in the region (1.10) by a function of time of the form

$$|\Psi(\alpha, \beta, \tau)| \leqslant C_{\Psi} \vartheta(\tau), \quad \vartheta(\tau) = \int_0^{\tau} |w_*(\tau')| (\tau - \tau')\, d\tau' \tag{3.1}$$

$$C_{\Psi}(r, \gamma) = 8\pi^{-2}\gamma (1 + e^{-1/2\gamma}) \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \chi_{in}(r, \gamma)$$

The dual series in estimate (3.1) converges for any $0 < r^2, \gamma^2 < \infty$ absolutely and uniformly, and can be easily estimated since its coefficients decay fairly rapidly (as the quantities $c_{in} \sim [in(i^2 + n^2)]^{-1}$).

The derivatives of the stream function $\psi = h^2\Psi$ in a, b determining the displacements ξ, η of the particles are obtained, according to (1.9), using term by term differentiation of the coefficients $\Phi_{mn}(\alpha, \beta)$ (2.3). The functions $\xi(a, b, t)$, $\eta(a, b, t)$ are majorized by the following functions of time (analogous to (3.1) for $\Psi(\alpha, \beta, \tau)$):

$$|\xi| \leqslant C_{\xi}\vartheta(\tau), \quad |\eta| \leqslant C_{\eta}\vartheta(\tau), \quad \tau = N_0 t \in [0, \Theta] \tag{3.2}$$

$$C_{\zeta} = 8\pi^{-2}\gamma(1 + e^{-1/2\gamma})\, h \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} D_{in}^{\zeta} \chi_{in} \quad (\zeta = \xi, \eta)$$

$$D_{in}^{\xi} = [n^2 + \gamma^2/(4\pi^2)]^{1/2}, \quad D_{in}^{\eta} = 2i + 1$$

The numerical binary series (3.2) for $C_{\zeta}(\zeta = \xi, \eta)$ converge, since their coefficients decrease more rapidly than $(in)^{-3/2}$.

Partial differentiation in t (or τ) does not impair the convergence of the series for

Ψ and ξ, η (see (2.5)), since the frequencies $\omega_{mn} \leqslant N_0$ (or $\omega_{mn} < 1$) are bounded. Therefore, expressions (2.5) for the velocities of the fluid particles are obtained by termwise differentiation of the Fourier coefficients $\theta_{in}(\tau)$. It is interesting to note that in the case the properties of summability of the series over n are improved, and the convergence in i is impaired (see below).

The convergence of the higher-order derivatives in α, β of the stream function $\Psi(\alpha, \beta, \tau)$ is not, in general, guaranteed. Therefore the solution $\Psi = \Psi(\alpha, \beta, \tau)$ of the boundary value problem (1.11), (1.12) should be regarded in the sense of strong convergence of Eq.(1.11) obtained by termwise differentiation, i.e. over the norm of the space $W_2^{(2)}(\Delta)$ with weight $e^{-\gamma\beta}$ /6, 7/.

4. Control of the motion of the vessel and internal waves in the fluid.

We shall consider the problems of kinematic control of the motion of the vessel: the acceleration $w(t) = dc(t)/dt$, $t \in [0, T]$ is regarded as a control belonging to the class of piecewise smooth functions in question, ensuring the existence of the solution of the internal boundary value problem (see Sect.3). A large number of formulations for the problems of controlling systems with distributed parameters and many definitions of the controllability appear in /8-12/ et al. Below we make use of the concept of controllability adopted in e.g. /8, 11, 12/, as the controllability for the system comprising an even number of pendulums by means of a single control function (see also /2, 3, 13, 14/).

4.1. From the practical point of view the solution of the problem in the case when the vessel is displaced from some initial state $s(0) = s^\circ$, $c(0) = c^\circ$ $(c = ds/dt)$ to a given final state $s(T) = s^*$, $c(T) = c^*$ (or to the state in which one of the quantities $s(T)$ or $c(T)$ is not fixed) is of interest. Here we require that the fluid should execute a given motion relative to the walls of the vessel, e.g. that it be at rest.

4.2. Using the properties of the spectrum $\{\omega_{mn}\}$ of the characteristic oscillations of the exponentially stratified fluid in a vessel, established in Sect.2, we find that in the general case the system in question, with a denumerable number of oscillatory degrees of freedom, cannot be controlled in a finite time interval /2, 3, 11, 12/. Also, in the case in question there is no asymptotic quasistationary solution of the problem related to the choice of a control $w(t)$, $t \in [0, T]$ of sufficiently small magnitude and a correspondingly large magnitude of the end of time interval T, such that the problem of the displacement of the vessel is solved and the displacements of the fluid particles remain asymptotically small (as in /2, 3/). The situation in question can be explained by the fact that the frequencies of natural oscillations $\omega_{mn} \downarrow 0$ as $n \to \infty$ and for a fixed value of $m \geqslant 1$. Therefore, when the variables $s(t)$ or $c(t)$ are changed by a signficant amount of the order of unity, so will the horizontal component of the displacement $\xi(a, b, t)$. It can be established that under such a control the quantities $\psi(a, b, t), \eta(a, b, t)$ and $v_x = \partial\xi/\partial t, v_y = \partial\eta/\partial t$ will remain asymptotically small. The control in question can be realized e.g. with help of the functions $w(t)$ of the following form.

1°. Piecewise constant functions $w(t)$, $t \in [0, T]$:

$$\begin{gathered} w(t) = w_k, \quad t \in (t_{k-1}, t_k], \quad w_k = \text{const}, \quad k = 1, \ldots, k^* \\ t_0 = 0, \quad t_{k^*} = T, \quad t_{k-1} < t_k, \quad \bigcup_{k=1}^{k^*} (t_{k-1}, t_k] = (0, T] \end{gathered} \tag{4.1}$$

Here the parameters of the control w_k, t_k, k^* have to be determined /2, 3/.

2°. The rapidly oscillating controls $w(t)$, $t \in (0, T]$:

$$\begin{gathered} w(t) = w_0 + \sum_{p=1}^{p^*} (w_p^c \cos \nu_p t + w_p^s \sin \nu_p t) \\ w_0, \quad w_p^{c,s}, \quad \nu_{p-1} < \nu_p = \text{const}, \quad \nu_p/N_0 > 1, \quad p = 1, \ldots, p^* \end{gathered} \tag{4.2}$$

The condition $\nu_p > N_0$ in (4.2) helps to avoid the resonance (see expression (2.3) for $\theta_{in}(\tau)$). The control parameters have to be determined from the final conditions.

The control $w(t)$, $t \in [0, T]$ can represent the sum of functions of the form (4.1), (4.2), etc. shown above.

The quantities $|w_k|$, $|w_0|$, $|w_p^{c,s}|$ in expressions (4.1), (4.2) are assumed to be sufficiently small, and the quantity T is assumed to be large, while the variations $s(T) - c^\circ T - s^\circ$ and (or) $c(T) - c^\circ$ must be significant (of the order of unity). All this is of practical importance.

In the special case of the control $w(t)$ (4.2) when $w_0 = 0$ and the mean value $\langle c(t)\rangle_t$ of the velocity $c(t)$ $(c^\cdot(t) = w(t), c(0) = c^\circ)$ is identical with the initial value c° $(\langle c(t)\rangle_t = c^\circ)$, we arrive at asymptotically small changes in the quantities $c(t)$ and $s(t)$ for all $t \in (0, T]$:

$c(t) - c^\circ = O(\max_t |w| \nu_1^{-1})$, $s(t) - c^\circ t - s^\circ = O(\max_t |w| \rangle \nu^{-2})$, and at asymptotically small quantities $\psi, \xi, \eta, v_x, v_y$. The last assertion follows from the formulas of the type (2.3), (2.5), and expressions for $\partial\psi / \partial b, -\partial\psi / \partial a$, resulting from elementary integration with respect to time and estimation of the series, which in this particular case converge absolutely and uniformly and are bounded uniformly in t for $t \in [0, T]$, $T < \infty$. If, on the other hand, the function $w(t)$ (4.2) brings about a significant change in $s(t)$ ($\langle c(t)\rangle_t \neq c^\circ$, for example, $w_0 \neq 0$), it would seem that such a control will lead to an analogous significant change in the value of $\xi = \xi(a, b, t)$, since we cannot establish the corresponding estimates uniform in t for ξ (the estimates cause the series for $\partial\psi / \partial b$ to diverge over the index n). Thus a continuously stratified heavy ideal fluid completely filling a rectangular vessel "does not react" to small, horizontal, high-frequency vibrations of the vessel.

4.3. A substantial change in the state of the fluid (within the framework of the linear approach), i.e. an appreciable horizontal and vertical "displacement" of its particles occurs, according to (2.3), (2.5) and the expressions for the derivatives of the stream function ψ in a, b, at frequencies ν_p lying within the resonance band $\{\nu_p\} \in (0, N_0)$. The more effective influence of the monochromatic (single frequency) oscillations of the vessel resulting in the greatest increase in t of the displacements $|\xi|$ and $|\eta|$, is realized at $\nu_p = N_0\omega_{1n^*}$, where the dimensionless frequencies ω_{mn} are found, according to (2.2), $m = 1$ and $n = n^*$ and n^* has to be determined. The value $m = 1$ $(i = 0)$ corresponds to the largest values of the coefficients Ψ_{in} in the expressions for the displacements $\xi = \partial\psi / \partial b, \eta = -\partial\psi / \partial a$, since they decrease monotonically as $m = 2i + 1$ increases.

It can be established that for fixed $r^2 > 0$ and sufficiently small $\gamma^2 > 0$, the extremal value $n^* = 1$. In the general case of arbitrary $0 < r^2$, $\gamma^2 < \infty$ the extremal values of the indices $n^* = n_\xi^*$ for ξ and $n^* = n_\eta^*$ for η can be found from the solution of the corresponding maximum problems. According to (2.3) and (4.2) (when $w_0 = 0$) and expressions for $\partial\psi / \partial b$, $\partial\psi / \partial a$, in order to determine the optimal indices η_ζ^* $(\zeta = \xi, \eta)$, we must maximize over $n = 1, 2, \ldots$ the coefficients

$$f_\zeta(n) = n[n^2 + \gamma^2/(4\pi^2)]^{-\lambda_\zeta}[r^{-2} + n^2 + \gamma^2/(4\pi^2)]^{-1/2} \qquad (4.3)$$
$$\lambda_\xi = 1/2,\ \lambda_\eta = 1$$

Standard methods of analysing functions of a single variable yield the required values of the indices

$$n_\zeta^* = [x_\zeta] \vee n_\zeta^* = [x_\zeta] + 1 \qquad (4.4)$$
$$x_\xi = (\gamma^2/(4\pi^2))^{1/4}(r^{-2} + \gamma^2/(4\pi^2))^{1/4}, \quad x_\eta = 2^{-1/4}x_\xi$$

The square brackets in expressions (4.4) denote the integer part of the positive number x_ζ. It is clear that when $x_\zeta < 1$ (if γ^2 is sufficiently small), $n_\zeta^* = 1$. If $x_\zeta \geqslant 1$, we take the values $n_\zeta^* = [x_\zeta]$ under the conditions that $f_\zeta([x_\zeta]) > f_\zeta([x_\zeta] + 1)$, and the values $n_\zeta^* = [x_\zeta] + 1$ $(\zeta = \xi, \eta)$ if we have the opposite inequalities.

A study of the simultaneous oscillation of the fluid and the vessel resting on an elastic support may be of practical interest, as well as other formulations of the problems of rigid-body dynamics where the body has cavities of rectangular and other shapes containing a heavy, ideal, continuously stratified fluid.

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Translated by L.K.

PMM U.S.S.R.,Vol.51,No.4,pp.462-467,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON TURBULENT BOUNDARY LAYER STRUCTURE*

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Flow in the turbulent boundary layer (BL) with Reynolds number $R \to \infty$ is studied by a joining asymptotic expansion method. A three-layered asymptotic BL structure is set up, which includes, besides the viscous boundary region and the velocity defect region, an intermediate region in which a balance of inertia forces, and pressure and turbulent friction forces takes place and which is responsible for flow separation under the influence of a disadvantageous pressure gradient.

A study of turbulent BL structure, based on the asymptotic analysis of an open set of Reynolds equations, has been the subject of a number of investigations. Early papers /1, 2/ essentially contain the known elements of this type of analysis. The paper by Yajnik /3/ was the first attempt at a systematic approach to the problem of constructing joined asymptotic expansions for averaged flow functions in a turbulent BL as $R \to \infty$. Further developments were made in /4-6/. In all these studies the structure of the turbulent BL, either with or without a pressure gradient, was established as a double layer: an inner (boundary) region and an outer region. In the first of these, flow is defined by a known Prandtl wall law which states that the sum of friction stresses caused by viscous action and turbulent pulses of velocity remains invariant across the whole zone. Flow in the outer region is described by the Kármán velocity defect law and represents a slightly perturbed potential flow close to the solid surface.

The possibility of a formal joining of the solutions for these regions is often seen as proof of the existence of an overlap region between them and a logarithmic velocity profile. The joining conditions also make it possible to find that the relative thickness of the velocity defect region is of the order of $1/\ln R$, and that the thickness of the layer at the wall is of the order of $\ln R/R$ /3/.

A more detailed examination of flow in the boundary region carried out below, however, shows that the two-layered flow diagram does not take place in reality. This diagram does not contain a region where the internal friction forces, the pressure gradient and inertia forces have the same order of magnitude as $R \to \infty$, i.e. just that region which, according to the Prandtl definition, is itself a BL. This, in particular, excludes the possibility of explaining the flow separation under a disadvantageous pressure gradient. Indeed, the flow in the velocity defect region to a first approximation is not susceptible to the action of friction forces and the flow in the wall-law region is not subject to the pressure gradient (since, for this to be so, the latter must have an unpracticably large values of the order of $R/\ln^3 R$).

In /7/, based on experimental observations, a law of the wake was introduced into the consideration, successfully linking the laws of the wall and the velocity defect. By this law the velocity profile in the BL is essentially dependent on the pressure gradient and changes so that, as it approaches the separation point, it adopts the shape of a profile in the wake. The law of the wake can therefore be considered as proof that the BL structure is not two-layered, i.e. the overlap region of the wall and velocity defect law does not exist in reality (even for flow without a pressure gradient) and, consequently, it is essential to

**Prikl.Matem.Mekhan.*,51,4,593-599,1987